Teleportation of a state in view of the quantum theory of measurement

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We give a description of the teleportation of an unknown quantum state which takes into account the action of the measuring device and manifestly avoids any reference to the postulate of the state vector collapse.

I. INTRODUCTION

In 1993, Bennett *et al.* [1] introduced the novel idea of teleporting a state of a quantum system from Alice to Bob using an auxiliary system shared by Alice and Bob. The first experimental implementations of this idea were reported by Boschi *et al.* [2], Bouwmeester *et al.* [3], and Furusawa *et al.* [4]. In all of these papers the justification of teleportation is based on three very different assumptions: the existence of EPR states, the *collapse* of the wave function after the measurement and the exchange of *classical information* from Alice to Bob.

The existence of EPR states follows directly form the linearity of quantum mechanics; it was verified in the last two decades by many experiments, see for example [5] and [6]. The collapse of the wave function after a measurement is not a consequence of quantum mechanics, it is an extra assumption that gives rise to many conceptual problems, for a review see [7] or the recent paper [8]. The exchange of *classical* information is a well-known phenomenon in everyday life, nevertheless, if quantum mechanics is a general description of the world, as we believe, one has to be able to explain the teleportation using only a *quantum language*.

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In this Letter we show that teleportation can be achieved by means of two suitable unitary transformations U and W acting on the states. The first, U, represents a Bell-state premeasurement, carried out by Alice on her pair of particles 1 and 2, which produces a superposition of four orthogonal entangled (EPR) states. The second operation, W, represents a measurement that allows Bob to change the state of his particle 3 into an identical copy of the (unknown) input state of particle 1. In the experiment of Furusawa *et al.* [4], the system is a single mode of an electromagnetic field, U is realized by a measurement of the quadratures performed by Alice and W by a linear displacement of the quadratures performed by Bob according to the result of the Alice measurement. Our analysis proves that this experiment constitutes evidence of teleportation *without the need of any extra assumption.* In the other two experiments cited above the system is the polarization degree of freedom of a light beam, only the first transformation is performed while the second is omitted. This leaves the total system in an entangled state, with Bob's photon potentially being in one of four states with equal probability. Hence one may be tempted to conclude that "...teleportation is successfully achieved, albeit only in a quarter of the cases", [3], [9]. However, this conclusion rests on the problematic projection postulate in that the mixed state of Bob's photon is interpreted as a classical mixture. As we will show, there is no evidence of successful teleportation if the projection postulate is *not* assumed.

The use of the collapse postulate in the teleportation scheme was already criticized by Motoyashi *et al.* [10]. These authors proposed a teleportation scheme based on a conservation law -type of an explanation of the nonlocal EPR correlations. In view of the difficulties with the so-called common cause explanations of the EPR problem, see [11] for details, we do not follow here the ideas presented in [10].

To simplify the exposition, we consider two level systems, such as spin $\frac{1}{2}$ systems, as already discussed by

Bennett *et al.* [1]. In the next section we give a description of teleportation using unitary transformations.

II. TELEPORTATION

We consider three spin $\frac{1}{2}$ systems $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$, with the associated Hilbert spaces $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathbb{C}^2$. Fix the unit vectors $|+\rangle = (1,0)$ and $|-\rangle = (0,1)$ in $\mathbb{C}^2$; we write $|\pm\rangle_i \in \mathcal{H}_i$ whenever we wish to emphasize that the vectors in question are elements of the Hilbert space $\mathcal{H}_i$, $i = 1,2,3$. The Bell basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ is defined as

$$\Psi^{\pm} = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle \pm |-\rangle|+\rangle),\ \Phi^{\pm} = \frac{1}{\sqrt{2}}(|+\rangle|+\rangle \pm |-\rangle|-\rangle),$$

where the tensor product of vectors is simply written as $\phi\Psi$.

The system $\mathcal{S}_1$ is initially in a vector state $\phi = a|+\rangle_1 + b|-\rangle_1$ (with $|a|^2 + |b|^2 = 1$), which is to be teleported to the system $\mathcal{S}_3$. To achieve this, $\mathcal{S}_2$ and $\mathcal{S}_3$ are prepared initially in the entangled state Ψ^-_{23}. Then $\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3$ is initially in the product state $\phi\Psi^-_{23}$.

To teleport the state ϕ of $\mathcal{S}_1$ to $\mathcal{S}_3$, the Bennett *et al.* [1] protocol requires performing a Bell state measurement on $\mathcal{S}_1 + \mathcal{S}_2$, that is, a measurement of an observable A_{12} with its simple eigenvectors given by the Bell basis $\Psi^{\pm}_{12}, \Phi^{\pm}_{12}$.

In our analyis of the interaction between the system and the measuring probe in the context of quantum mechanics we use the notion of *premeasurement* [12,13]. Let $\mathcal{H}_0$ be the Hilbert space of the measuring probe $\mathcal{S}_0$. Since the probe must have four orthogonal indicator states corresponding to the eigenvectors $\Psi^{\pm}_{12}, \Phi^{\pm}_{12}$, one can assume that $\mathcal{H}_0 = \mathbb{C}^4$. Let η be the initial vector state of the probe, and let $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ be a fixed basis of $\mathcal{H}_0$. One can prove [12] that the most general measurement interaction between $\mathcal{S}_1 + \mathcal{S}_2$ and the probe is given by a unitary operator $U : \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$ of the form

$$\begin{aligned} U(\eta\Psi^+) &= \eta_1\chi_1 \qquad & U(\eta\Psi^-) &= \eta_2\chi_2 \\ U(\eta\Phi^+) &= \eta_3\chi_3 \qquad & U(\eta\Phi^-) &= \eta_4\chi_4, \end{aligned}$$

where χ_i are four unit vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$. The choice $\chi_1 = \Psi^+$, $\chi_2 = \Psi^-$, $\chi_3 = \Phi^+$, $\chi_4 = \Phi^-$ corresponds to a von Neumann-Lüders measurement.

Adopting this description, the state of the system and the probe after the measurement is

$$\begin{aligned} &(U \otimes I_3)(\eta\phi\Psi^-_{23}) \\ &\quad = \frac{1}{2}\eta_1\chi_1\phi_1 + \frac{1}{2}\eta_2\chi_2(\phi_2) + \frac{1}{2}\eta_3\chi_3\phi_3 + \frac{1}{2}\eta_4\chi_4\phi_4, \end{aligned}$$

where we have used the notations

$$\begin{aligned} &\phi_1 := -a|+\rangle + b|-\rangle,\ \phi_2 := -\phi, \\ &\phi_3 := a|-\rangle - b|+\rangle,\ \phi_4 := a|-\rangle + b|+\rangle. \end{aligned}$$

Since the vector state $(U \otimes I_3)(\eta\phi\Psi^-_{23})$ is entangled, the (reduced) states of the systems $\mathcal{S}_i$ are not vector states but mixed states represented by the following density operators:

$$\begin{aligned} T^f_0 &= \frac{1}{4}P[\eta_1] + \frac{1}{4}P[\eta_2] + \frac{1}{4}P[\eta_3] + \frac{1}{4}P[\eta_4] = \frac{1}{4}I_0, \\ T^f_3 &= \frac{1}{4}P[\phi_1] + \frac{1}{4}P[\phi] + \frac{1}{4}P[\phi_3] + \frac{1}{4}P[\phi_4] = \frac{1}{2}I_3, \\ T^f_{12} &= \frac{1}{4}P[\chi_1] + \frac{1}{4}P[\chi_2] + \frac{1}{4}P[\chi_3] + \frac{1}{4}P[\chi_4], \\ T^f_{012} &= \frac{1}{4}\sum_{i,j=1}^{4} \langle\phi_i, \phi_j\rangle\, |\eta_i\chi_i\rangle\langle\eta_j\chi_j| \\ T^f_{123} &= \frac{1}{4}P[\chi_1\phi_1] + \frac{1}{4}P[\chi_2\phi] + \frac{1}{4}P[\chi_3\phi_3] + \frac{1}{4}P[\chi_4\phi_4], \end{aligned}$$

where $P[\xi]$ denotes the projection on the one dimensional space generated by the unit vector ξ. We note that the mixed states T^f_0 and T^f_3 are completely unpolarized.

It is important to realize that it is inconsistent to interpret the state T^f_3 as a classical mixture of its component vector states ϕ_1, $\phi_2 = -\phi$, ϕ_3, ϕ_4. Such an *ignorance interpretation* is incompatible with the pure nature of the entangled total state $(U \otimes I_3)(\eta\phi\Psi^-_{23})$ of system plus probe. Although this crucial point has been known for many years, see, e.g., the classic analysis of d'Espagnat [14], it is frequently ignored in the discussion of experiments involving entanglement (for a notable recent exception in the context of teleportation, see [15]). In fact the EPR experiment itself can be taken as a demonstration of the inadmissability of the ignorance interpretation for the reduced mixed states, cf., e.g., [7]. Formally, this phenomenon is reflected in the fact that the decomposition of a mixed state into vector (or pure) states is never unique (for a full analysis, see e.g. [16]).

The same remarks apply to the reduced state T^f_0 of the probe, leading to the objectification problem, that is, the impossibility of explaining *within quantum mechanics* the occurrence of definite outcomes as a result of the measurement, represented by the appearance of one of the possible indicator states η_i. As is generally acknowledged, this problem constitutes a serious challenge to the view of quantum mechanics as a consistent and universal theory. The commonly accepted, pragmatic way of dealing with it is the application of the collapse postulate. While this seems good for (almost) all practical purposes in the discussion of most experiments, we hope the above remarks serve as a reminder of the fundamental inconsistency of such an approach.

It follows that after the premeasurement U the state of $\mathcal{S}_3$ is T^f_3, and this state yields no evidence of quantum teleportation as can be seen by computing the fidelity,

$$\mathrm{tr}\left(P[\phi]T^f_3\right) = \frac{1}{2}.$$

It is only when the collapse is postulated that one can conclude that "teleportation is successfully achieved ... in a quarter of the cases."

However, if the Bell state premeasurement U is followed by another unitary operation W, which models the classical communication and ensuing manipulation of $\mathcal{S}_3$, then the collapse postulate can be avoided, thus bypassing the objectification problem.

This final step consists of a rotation of the state of system $\mathcal{S}_3$ depending on the result of the previous measurement. Since the probe is itself a quantum system, we can model this conditional manipulation based on a classical information transfer as another premeasurement performed on the system $\mathcal{S}_0 + \mathcal{S}_3$ by means of a unitary operator $W : \mathcal{H}_0 \otimes \mathcal{H}_3 \to \mathcal{H}_0 \otimes \mathcal{H}_3$,

$$\begin{array}{ll} W(\eta_1|+\rangle) = -\eta_1|+\rangle & W(\eta_1|-\rangle) = \eta_1|-\rangle \\ W(\eta_2|+\rangle) = -\eta_2|+\rangle & W(\eta_2|-\rangle) = -\eta_2|-\rangle \\ W(\eta_3|+\rangle) = -\eta_3|-\rangle & W(\eta_3|-\rangle) = \eta_3|+\rangle \\ W(\eta_4|+\rangle) = \ \ \eta_4|-\rangle & W(\eta_4|-\rangle) = \eta_4|+\rangle \end{array}$$

In doing so, there is no need to account for the actual occurrence of a result, and the objectification problem is circumvented. Regarding W as a linear transformation on $\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ in a natural way, one obtains the final state

$$W\left(U(\eta\phi\Psi^-)\right) = \frac{1}{2}(\eta_1\chi_1 + \eta_2\chi_2 + \eta_3\chi_3 + \eta_4\chi_4)\phi.$$

The (reduced) state of the system $\mathcal{S}_3$ is now the vector state ϕ, which has been teleported with 100% efficiency from the system $\mathcal{S}_1$ to $\mathcal{S}_3$.

We note that since U and W are unitary operators, they can be realized (at least theoretically) by suitable quantum Hamiltonians.

In order to highlight the difference between our approach and the ones based on the projection postulate, we modify slightly the form of W. We define a unitary transformation W_θ (with $\theta \in [0, 2\pi)$) in such a way that it is equal to W save that

$$W_\theta(\eta_2|+\rangle) = -e^{i\theta}\eta_2|+\rangle.$$

A simple calculation shows that

$$\begin{aligned} W_\theta\left(U(\eta\phi\Psi^-)\right) &= \frac{1}{2}(\eta_1\chi_1 + \eta_3\chi_3 + \eta_4\chi_4)\phi + \frac{1}{2}\eta_2\chi_2\phi_\theta, \\ \phi_\theta &= (e^{i\theta}a|+\rangle + b|-\rangle), \end{aligned}$$

and the final reduced state of $\mathcal{S}_3$ is now

$$T_3^f(\theta) = \frac{3}{4}P[\phi] + \frac{1}{4}P[\phi_\theta],$$

If $\theta \neq 0$, $a \neq 0$, and $b \neq 0$, then $T_3^f(\theta)$ is a mixed state different from the pure state $P[\phi]$, so that, without the collapse assumption or ignorance interpretation, there is no teleportation. Nevertheless, in the limit $\theta \to 0$, we have $\phi_\theta \to \phi$, and so $T_3^f(\theta)$ continuously approaches $P[\phi]$, thus leading to perfect teleportation upon reaching $\theta = 0$. By contrast, if one accepts the projection postulate, one could say that the probe collapses into one of the four states $\eta_1, \ldots, \eta_4$, each of them corresponding to a definite value of the measurement, and, performing a coincidence detection, one could say that $\mathcal{S}_3$ is found in the exact target state ϕ in 75% of the cases. In doing so one has to face the fact that, in the limit $\theta \mapsto 0$, the rate of *correct* target states [successful perfect teleportations] jumps from 75% to 100%; this fact contrasts with the common understanding that Nature is *continuous* with respect the variation of *parameters* such as θ. It is interesting to observe that the teleportation fidelity, being a statistical measure, cannot distinguish between the two interpretations of $T_3^f(\theta)$ (a) as a quantum mixed state – in which case one can speak of *approximate* teleportation as $\theta \to 0$, and (b) as a classical mixture – in which case there is perfect teleportation in 75% of the cases. In either case, the fidelity is $\mathrm{tr}\left(P[\phi]T_3^f(\theta)\right)$. Nevertheless this does not justify the conclusion that the two approaches are physically equivalent.

Finally we note that, in our approach, the coincidence detection can be simply interpreted as a measurement of the observable (with possible outcomes 0 and 1) on $\mathcal{H}_0 \otimes \mathcal{H}_3$

$$A = P[\eta_1\phi] + P[\eta_3\phi] + P[\eta_4\phi].$$

Indeed, if we regard A in a natural way as operator on $\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, we have that

$$\langle W_\theta\left(U(\eta\phi\Psi^-)\right)|A|W_\theta\left(U(\eta\phi\Psi^-)\right)\rangle = \frac{3}{4}.$$

III. CONCLUSION

We have given a measurement theoretical description of the teleportation of an unknown quantum state which takes into account the action of the measuring device and manifestly avoids any reference to the state vector collapse postulate. The teleportation process can be described as a measurement whose outcome is certain; equivalently the process is realized by the action of a unitary map. In other words, teleportation can be carried out as a deterministic process and hence does not require stochastic (or quantum) jumps in the sense expressed by the collapse postulate.

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